

Geometric Thermodynamics of Kerr-AdS Black Hole with the Cosmological Constant as a State Variable

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Abstract: The thermodynamics of the Kerr-AdS black hole is reformulated within the context of the formalism of geometrothermodynamics (GTD) and the cosmological constant is considered as a thermodynamical parameter. We conclude that the mass of the black hole corresponds to the total enthalpy of this system. Choosing appropriately the metric in the manifold of equilibrium states, we study the phase transitions as a divergence of the thermodynamical curvature scalar. This approach reproduces the Hawking-Page transition and shows that considering the cosmological constant as a thermodynamical parameter does not contribute new phase transitions to the pre-existing picture.

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§1. Introduction. The thermodynamics of black holes has been studied extensively since the work of Hawking [1]. The notion of critical behavior for black holes has arisen in several contexts from the Hawking-Page [2] phase transition in anti-de-Sitter (AdS) background to the pioneering work by Davies [3] on the thermodynamics of Kerr-Newman black holes and the idea of the extremal limit of various black hole families regarded as genuine critical points [4–6]. Recently, some authors have considered the cosmological constant Λ as a dynamical variable [7, 8] and it has further been suggested that it is better to consider Λ as a thermodynamic variable, [9–13]. Physically, Λ is interpreted as a thermodynamic pressure in [14, 15], a fact that is consistent with the observation in [16–18] that its conjugate thermodynamic variable is proportional to a volume.

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The use of geometry in statistical mechanics was pioneered by Ruppeiner [19] and Weinhold [20], who suggested that the curvature of a metric defined on the space of parameters of a statistical mechanical theory could provide information about its phase structure. When this treatment is applied to the study of black hole thermodynamics, some puzzling anomalies appear. A possible solution was suggested by Quevedo's geometrothermodynamics (GTD) whose starting point [21] was the observation that standard thermodynamics is invariant with respect to Legendre transformations. The formalism of GTD indicates that phase transitions occur at those points where the thermodynamic curvature scalar is singular.

In this paper, we apply the GTD formalism to the Kerr-AdS black hole to investigate the behavior of the thermodynamical curvature. As is well known, a black hole with a positive cosmological constant has both a cosmological horizon and an event horizon. These two surfaces have, in general, different Hawking temperatures, which complicates any thermodynamical treatment. Therefore, we will focus on the case of a negative cosmological constant, though many of the conclusions are applicable to the positive Λ case. Furthermore, the negative Λ case is of interest for studies on AdS/CFT correspondence and the subsequent considerations of this work are likely to be relevant in those studies.

§2. Geometrothermodynamics in brief. The formulation of GTD is based on the use of contact geometry as a framework for thermodynamics. The $(2n + 1)$ -dimensional thermodynamic phase space \mathcal{T} is coordinatized by the thermodynamic potential Φ , the extensive variables E^a , and the intensive variables I^a , with $a = 1, \dots, n$. We define on \mathcal{T} a non-degenerate metric $G = G(Z^A)$ with $Z^A = \{\Phi, E^a, I^a\}$, and the Gibbs 1-form $\Theta = d\Phi - \delta_{ab} I^a dE^b$ with $\delta_{ab} = \text{diag}(1, 1, \dots, 1)$. If the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied, the set (\mathcal{T}, Θ, G) defines a contact Riemannian manifold. The Gibbs 1-form is invariant with respect to Legendre transformations, while the metric G is Legendre invariant if its functional dependence on Z^A does not change under a Legendre transformation. This invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction.

Now, we define the n -dimensional subspace of equilibrium thermodynamic states, $\mathcal{E} \subset \mathcal{T}$, by means of the smooth mapping

$$\left. \begin{aligned} \varphi : \mathcal{E} &\longrightarrow \mathcal{T} \\ (E^a) &\longmapsto (\Phi, E^a, I^a) \end{aligned} \right\}, \quad (1)$$

with $\Phi = \Phi(E^a)$, and the condition $\varphi^*(\Theta) = 0$, which gives the first law of thermodynamics

$$d\Phi = \delta_{ab} I^a dE^b, \quad (2)$$

and the conditions for thermodynamic equilibrium (the intensive thermodynamic variables are dual to the extensive ones),

$$\frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b. \quad (3)$$

The mapping φ defined above implies that we know the equation $\Phi = \Phi(E^a)$ explicitly. It is known as the fundamental equation, and from it can be derived all the equations of state. The second law of thermodynamics is equivalent to the convexity condition on the thermodynamic potential,

$$\frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0. \quad (4)$$

Since the thermodynamic potential satisfies the homogeneity condition $\Phi(\lambda E^a) = \lambda^\beta \Phi(E^a)$ for constant parameters λ and β , it satisfies Euler's identity,

$$\beta \Phi(E^a) = \delta_{ab} I^b E^a, \quad (5)$$

and using the first law of thermodynamics, this gives the Gibbs-Duhem relation,

$$(1 - \beta) \delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0. \quad (6)$$

Defining a non-degenerate metric structure g on \mathcal{E} that is compatible with a metric G on \mathcal{T} , we state that a thermodynamic system is described by the thermodynamical metric G [21] if it is invariant with respect to transformations which do not modify the contact structure of \mathcal{T} . In particular, G must be invariant with respect to Legendre transformations in order for GTD to be able to describe thermodynamic properties in terms of geometric concepts independently of the the thermodynamic potential used. A partial Legendre transformation is written as

$$Z^A \rightarrow \tilde{Z}^A = \left\{ \tilde{\Phi}, \tilde{E}^a, \tilde{I}^a \right\}, \quad (7)$$

where

$$\left. \begin{aligned} \Phi &= \tilde{\Phi} - \delta_{kl} \tilde{E}^k \tilde{I}^l \\ E^i &= -\tilde{I}^i \\ E^j &= \tilde{E}^j \\ I^i &= \tilde{E}^i \\ I^j &= \tilde{I}^j \end{aligned} \right\}, \quad (8)$$

with $i \cup j$ any disjoint decomposition of the set of indices $\{1, 2, \dots, n\}$ and $k, l = 1, \dots, i$. As is shown in [21], a Legendre invariant metric G induces a Legendre invariant metric g on \mathcal{E} defined by the pullback φ^* as $g = \varphi^*(G)$. There is a vast number of metrics on \mathcal{T} that satisfy the Legendre invariance condition. The results of Quevedo et al. [22–24] show that phase transitions occur at those points where the thermodynamic curvature is singular and that the metric structure of the phase manifold \mathcal{T} determines the type of systems that can be described by a specific thermodynamic metric. For instance, a pseudo-Euclidean structure

$$G = \Theta^2 + (\delta_{ab} E^a I^b) (\eta_{cd} dE^c dI^d) \quad (9)$$

with $\eta_{cd} = \text{diag}(-1, 1, 1, \dots, 1)$ is Legendre invariant because of the invariance of the Gibbs 1-form and induces on \mathcal{E} the Quevedo's metric

$$g = \left(E^f \frac{\partial \Phi}{\partial E^f} \right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d \right), \quad (10)$$

which describes systems characterized with second-order phase transitions. On the other hand, an Euclidean structure

$$G = \Theta^2 + (\delta_{ab} E^a I^b) (\delta_{cd} dE^c dI^d) \quad (11)$$

is also a Legendre invariant and induces on \mathcal{E} the metric

$$g = \left(E^f \frac{\partial \Phi}{\partial E^f} \right) \left(\frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^c dE^d \right), \quad (12)$$

which describes systems with first-order phase transitions.

§3. The Kerr-AdS black hole. The Einstein action with cosmological constant Λ is given by

$$\mathcal{A} = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda), \quad (13)$$

and the general solution representing a black hole is given by the Kerr-AdS solution

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\varphi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi} d\varphi \right)^2 + \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right), \quad (14)$$

where

$$\Delta_r = (r^2 + a^2) \left(1 - \frac{\Lambda r^2}{3} \right) - 2mr, \quad (15)$$

$$\Delta_\theta = 1 + \frac{\Lambda a^2}{3} \cos^2 \theta, \quad (16)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad (17)$$

and

$$\Xi = 1 + \frac{\Lambda a^2}{3}. \quad (18)$$

The physical parameters of the black hole can be obtained by means of Komar integrals using the Killing vectors $\frac{\partial t}{\Xi}$ and ∂_φ . In this way, one obtains the mass of the black hole

$$M = \frac{m}{\Xi^2} \quad (19)$$

and its angular momentum

$$J = aM = a \frac{m}{\Xi^2}. \quad (20)$$

The horizons are given by the roots of

$$\Delta_r = 0. \quad (21)$$

In particular, the largest positive root located at $r = r_+$ defines the event horizon with an area

$$A = 4\pi \frac{(r_+^2 + a^2)}{\Xi}. \quad (22)$$

The Smarr formula for the Kerr-AdS black hole gives the relation

$$M^2 = J^2 \left(\frac{\pi}{S} - \frac{\Lambda}{3} \right) + \frac{S^3}{4\pi^3} \left(\frac{\pi}{S} - \frac{\Lambda}{3} \right)^2 \quad (23)$$

that corresponds to the fundamental thermodynamical equation $M = M(S, J, \Lambda)$ which relates the total mass M of the black hole with the extensive variables, entropy $S = \frac{A}{4}$, angular momentum J and cosmological constant Λ , and from which all the thermodynamical information can be derived.

In the geometric formulation of thermodynamics, we will choose the extensive variables as $E^a = \{S, J, \Lambda\}$ and the corresponding intensive

variables as $I^a = \{T, \Omega, \Psi\}$, where T is the temperature, Ω is the angular velocity and Ψ is the generalized variable conjugate to the state parameter Λ . Therefore, the coordinates that we will use in the 7-dimensional thermodynamical space \mathcal{T} are $Z^A = \{M, S, J, \Lambda, T, \Omega, \Psi\}$. The contact structure of \mathcal{T} is generated by the 1-form

$$\Theta = dM - TdS - \Omega dJ - \Psi d\Lambda. \quad (24)$$

To obtain the induced metric in the space of equilibrium states \mathcal{E} we will introduce the smooth mapping

$$\begin{aligned} \varphi : \{S, J, \Lambda\} &\longmapsto \\ &\longmapsto \{M(S, J, \Lambda), S, J, \Lambda, T(S, J, \Lambda), \Omega(S, J, \Lambda), \Psi(S, J, \Lambda)\} \end{aligned} \quad (25)$$

along with the condition $\varphi^*(\Theta) = 0$, that corresponds to the first law $dM = TdS + \Omega dJ + \Psi d\Lambda$. This condition also gives the relation between the different variables with the use of the fundamental relation (23). The Hawking temperature is evaluated as

$$T = \frac{\partial M}{\partial S} = \frac{S^2}{8\pi^3 M} \left(\frac{\pi}{S} - \frac{\Lambda}{3} \right) \left(\frac{\pi}{S} - \Lambda \right) - \frac{\pi J^2}{2MS^2}, \quad (26)$$

the angular velocity is

$$\Omega = \frac{\partial M}{\partial J} = \frac{J}{M} \left(\frac{\pi}{S} - \frac{\Lambda}{3} \right) \quad (27)$$

and the conjugate variable to Λ is

$$\Psi = \frac{\partial M}{\partial \Lambda} = -\frac{S^3}{12\pi^3 M} \left(\frac{\pi}{S} - \frac{\Lambda}{3} \right) - \frac{J^2}{6M}. \quad (28)$$

As can be seen, Ψ has dimensions of a volume. In fact, in the limit of a non-rotating black hole, $J \rightarrow 0$, we have $\Psi = -\frac{4}{3}r_+^3$ (see [25]) and it can be interpreted as an effective volume excluded by the horizon, or alternatively a regularized version of the difference in the total volume of space with and without the black hole present [14–16]. Since the cosmological constant Λ behaves like a pressure and its conjugate variable as a volume, the term $\Psi d\Lambda$ has the correct dimensions of energy and is the analogue of VdP in the first law. This suggests that after expanding the set of thermodynamic variables to include the cosmological constant, the mass M of the AdS black hole should be interpreted as the *enthalpy* rather than as the total energy of the spacetime.

The \mathcal{T} becomes a Riemannian manifold by defining the metric (9),

$$G = (dM - TdS - \Omega dJ - \Psi d\Lambda)^2 + (ST + \Omega J + \Psi \Lambda)(-dSdT + dJd\Omega + d\Lambda d\Psi). \quad (29)$$

The G has non-zero curvature and its determinant is $\det \|G\| = -\frac{(ST + \Omega J + \Psi \Lambda)^6}{64}$. Equation (10) lets us define the induced metric structure on \mathcal{E} as

$$g = (SM_S + JM_J + \Lambda M_\Lambda) \begin{pmatrix} -M_{SS} & 0 & 0 \\ 0 & M_{JJ} & M_{J\Lambda} \\ 0 & M_{J\Lambda} & M_{\Lambda\Lambda} \end{pmatrix}, \quad (30)$$

where subscripts represent partial differentiation with respect to the corresponding coordinate. Note that the determinant of this metric is

$$\det \|g\| = M_{SS} (M_{J\Lambda}^2 - M_{JJ}M_{\Lambda\Lambda}) (SM_S + JM_J + \Lambda M_\Lambda)^3. \quad (31)$$

We can also define an Euclidean metric (11) on \mathcal{T} , but there are no phase transitions associated with this metric.

§4. Phase transitions and the curvature scalar. Phase transitions are an interesting subject in the study of black hole thermodynamics since there is no unanimity in their definition. In ordinary thermodynamics, phase transitions are defined by looking for singular points in the behavior of thermodynamical variables. Davis [3,26] shows that the divergences in the heat capacity indicate phase transitions. For example, using equation (23) we have that the heat capacity for the Kerr-AdS black hole is

$$C = T \frac{\partial S}{\partial T} = \frac{M_S}{M_{SS}}, \quad (32)$$

$$C = \frac{S \left(\frac{\pi}{S} - \frac{\Lambda}{3} \right) \left(\frac{\pi}{S} - \Lambda \right) - \frac{4\pi^4 J^2}{S^3}}{\left(\frac{\pi}{S} - \frac{\Lambda}{3} \right) \left(\frac{\pi}{S} - 2\Lambda \right) - \frac{\pi}{S} \left(\frac{\pi}{S} - \Lambda \right) + \frac{8\pi^3}{S^2} \left(\frac{\pi J^2}{S^2} - ST^2 \right)}. \quad (33)$$

Thus, one can expect that phase transitions occur at the divergences of C , i.e. at points where $M_{SS} = 0$. For negative Λ the divergence of C corresponds to the generalization of the well-known Hawking-Page transition [2]. In GTD, the emergence of phase transitions appears to be related with the divergences of the curvature scalar R in the space of equilibrium states \mathcal{E} . To understand this relation, remember that R

$$M_{SS} = \frac{144\pi^7 J^4 (9\pi - 4\Lambda S) + 24\pi^3 J^2 S^2 (3\pi - 2\Lambda S) (\Lambda S - 3\pi)^2 + S^4 (\Lambda S - 3\pi)^3 (\Lambda S + \pi)}{8\pi^{3/2} S^4 \left[\frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}},$$

$$M_{JJ} = -\frac{2\pi^{3/2} (\Lambda S - 3\pi)^3}{\left[\frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}},$$

$$M_{\Lambda\Lambda} = -\frac{6\pi^{9/2} J^4}{\left[\frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}},$$

$$M_{J\Lambda} = \frac{12\pi^{9/2} J^3 (\Lambda S - 3\pi)}{S \left[\frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}}.$$

always contains the determinant of the metric g in the denominator and, therefore, the zeros of $\det \|g\|$ could lead to curvature singularities (if those zeros are not cancelled by the zeros of the numerator).

Here we have considered the metric g given in (30) and its determinant is proportional to M_{SS} as shown in equation (31), making clear the coincidence with the divergence of the heat capacity and the existence of a second-order phase transition that corresponds to the generalization of the Hawking-Page result. There is also a factor of $(M_{J\Lambda}^2 - M_{JJ}M_{\Lambda\Lambda})$ in the determinant which codifies the information of non-constant Λ . Note that the interesting second derivatives of the thermodynamic potential are shown in Page 75.

As can be seen, for negative values of Λ the factor $(M_{J\Lambda}^2 - M_{JJ}M_{\Lambda\Lambda})$ is always positive. Therefore, we conclude that considering Λ as a new thermodynamical state parameter does not produce new phase transitions in the Kerr-AdS black hole.

§5. Conclusion. Quevedo's geometrothermodynamics describes in an invariant manner the properties of thermodynamic systems using geometric concepts. It indicates that phase transitions would occur at those points where the thermodynamic curvature R is singular. Following Quevedo, the choice of the metric given in equation (10) apparently describes second-order phase transitions.

In this work, we have applied the GTD formalism to the Kerr-AdS black hole, considering the cosmological constant as a new thermodynamical state variable. In this approach, the total mass of the black hole is interpreted as the total enthalpy of the system. Thus, we have obtained a curvature scalar that diverges exactly at the point where the Hawking-Page phase transition occurs. Since we have employed a metric of the form given in (10) we conclude that this is a second-order phase transition. It is also important to note that the consideration of Λ as a thermodynamical variable does not include new phase transitions in the system.

It is clear that the phase manifold in the GTD formalism contains information about thermodynamic systems; however, it is not clear at present where the thermodynamic information is encoded. A more detailed investigation along these lines will be reported in the future.

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